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SHAPE FACTORS IN CONDUCTION HEAT FLOW FOR CIRCULAR BARS AND SLABS WITH VARIOUS INTERNAL GEOMETRIES

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Abstract—Shape factors describing the steady-state, two-dimensional flow of heat in circular bars and slabs with various internal geometries and isothermal boundary conditions are computed by a conformal mapping procedure.

NOMENCLATURE

A,	conducting area;
B,	thickness of a slab;
D,	outside diameter of a bar,
L,	axial dimension of a bar or slab;
Q,	heat flow rate;
T,	temperature at a point (x, y);
T_i,	temperature at the radial slits in the Z-plane, temperature at the inner boundary ρ_i in the W-plane;
T_o,	temperature at the outer boundary in both W- and Z-planes;
T_p,	temperature at the polygon shaped central hole in the Z-plane;
W,	complex number in the ρ, θ plane;
W',	complex number in the ρ', θ' plane;
Z,	complex number in the γ, ϕ plane;
Z',	complex number in the γ', ϕ' plane;
d,	diameter of the inscribed circle of a polygon;
f,	semi focal slit in the τ-plane;
k,	thermal conductivity;
l,	length of a radial slit;
n,	number of radial slits, number of sides of polygon;
r,	radius of the "common circle" isotherm in the Z-plane;
x, y,	Cartesian co-ordinates of a point.

Greek symbols

ρ, θ, ρ', θ', γ, ϕ, γ', ϕ',	polar co-ordinates of a point;
τ, τ',	complex numbers in the intermediate mapping planes.

Subscripts

0,	outer boundary;
i,	radial slits;
p,	polygonal hole.

INTRODUCTION

FOR CALCULATING the steady conduction heat flow between two surfaces shape factors are widely used; however, there are many complicated shapes for which no analytic solutions for these factors are available. In particular the solution of configurations with polygon boundaries is difficult. Balcerzak and Raynor [1] have derived approximate expressions for prismatic bars with a central circular hole but solutions remain to be presented for the complementary configuration of a circular bar with a central polygon hole. For this case the electric analogue method is used by Murthy and Ramachandran [2]; their empirical solutions are limited, however, to circular bars with either a hexagonal or a square hole only. With the electric analogue technique inaccuracies can occur through the difficulty of providing a uniform potential along each boundary of the model and because the conducting paper resistance is not everywhere uniform.

Analytic solutions for the shape factor of long circular bars with a single central polygon hole are now derived by a procedure based on the well-known conformal mapping technique. The method also yields solutions for bars with central radial slits and it is extended to include expressions for semi-infinite slabs. The configurations which are analysed and their shape factors are given in Table 1.

The flow is due to a heat source along the axis of the hole maintaining uniform surface temperatures. The material is assumed to be homogeneous, isotropic with temperature independent properties.

In the analysis the principle is invoked that three or more straight isotherms, of equal length and radiating at constant intervals from a common point in a Laplacian field generate an isotherm which is a regular polygon with rounded corners beyond which a boundary isotherm is circular about the common point—Fig. 1.

THEORETICAL BASIS

The temperature distribution in the sections satisfy the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

where T is the temperature at a point whose co-ordinates are x and y. A solution of this equation satisfying the boundary conditions of a section will determine the temperature distribution and pattern of heat flow. This is achieved by transforming the complicated geometry in the complex Z-plane to a simple shape in the complex W-plane by conformal mapping procedure. A point (x, y) in the Z-plane is to be mapped into a corresponding point (ρ, θ) in the W-plane. This is accomplished by the conformal transformation

$$W = F(Z)$$

where $F(Z)$ is regular and $F'(Z)$ is non-zero. The temperature T satisfying the Laplace equation in the Z-plane will do so in the W-plane also [3] and there will exist corresponding points in the two planes having identical temperatures for equivalent boundary conditions.

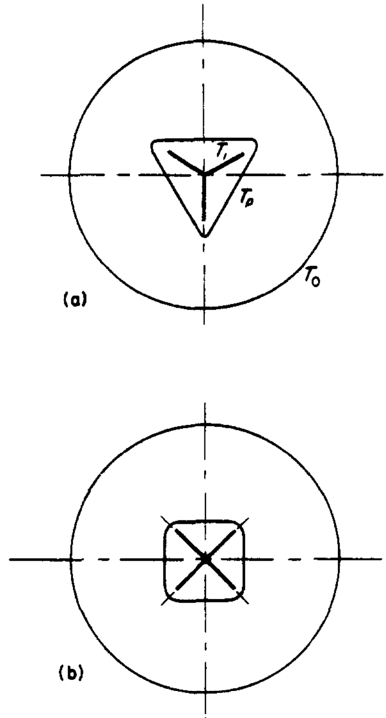


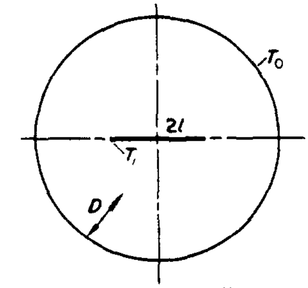
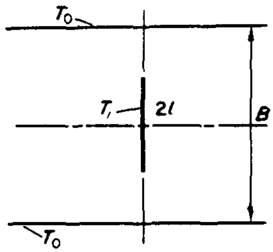
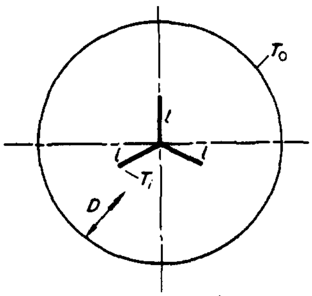
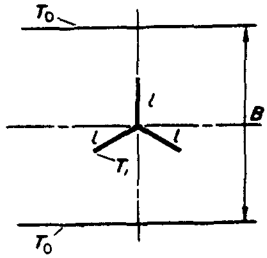
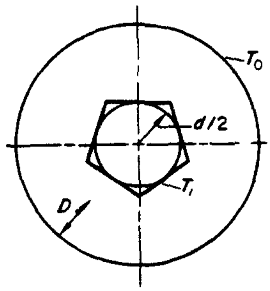
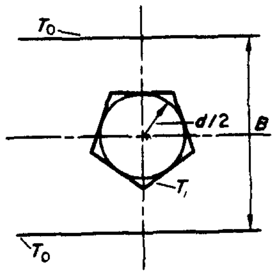
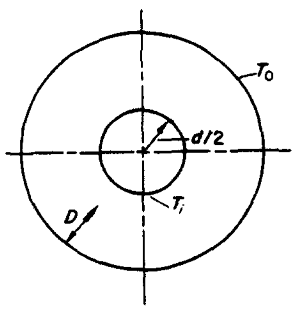
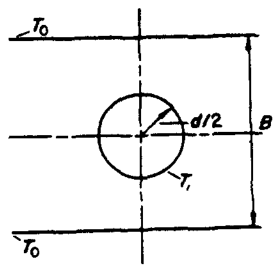
FIG. 1. Generation of polygon isotherms.
(a) Equilateral triangular isotherm.
(b) Square isotherm.

In the W-plane the simple shape which will be used is the circular bar with a small central circular hole along its axis. The flow is radial, in the ρ direction, with the boundary conditions

$$T = T_0 \quad \text{at} \quad \rho = \rho_0 = 1$$
$$\text{and } T = T_i \quad \text{at} \quad \rho = \rho_i.$$

The Laplace equation in cylindrical co-ordinates

Table 1. Summary of results

Long circular bar		Semi-infinite slab	
shape and notation	shape factor $Q/[Lk(T_i - T_0)]$	shape and notation	shape factor $Q/[Lk(T_i - T_0)]$
 Single central slit	$2\pi/\ln (D/l)$	 Single central slit	$2\pi/\ln\left(\frac{4B}{\pi l}\right)$
 n equi-pitched radial slits	$2\pi/\ln (A_1 D/l)$ Values of A_1 from Table 2	 n equi-pitched radial slits	$2\pi/\ln\left\{\frac{2^{(2+n)/n}B}{\pi l}\right\}$
 n sided regular polygon hole	$2\pi/\ln (A_2 D/d)$ Values of A_2 from Table 2	 n sided regular polygon hole	$2\pi/\ln\left\{\frac{4B}{\pi d}\left[\frac{2}{\sqrt{\left(\frac{n}{n-2}\right)+1}}\right]^{2/n}\right\}$
 Central circular hole	$2\pi/\ln (D/d)$	 Central circular hole	$2\pi/\ln\left(\frac{4B}{\pi d}\right)$

is then reduced for this one-dimensional, radial flow in the W-plane to

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial T}{\partial \rho}\right) = 0$$

which on integration gives, with the boundary conditions,

$$T - T_0 = (T_i - T_0) \ln \rho / \ln \rho_i \qquad (1)$$

for the temperature distribution where T is the temperature at radius ρ which is less than the unit radius at the outer circular boundary.

The heat flow is derived from the one-dimensional Fourier's law in cylindrical form

$$Q = -kA\,\mathrm{d}T/\mathrm{d}\rho$$

where now the conducting area A is a function of the radius ρ

$$A = 2\pi\rho L.$$

By separating variables and integrating between the inner and outer boundaries there follows

$$Q = -2\pi kL(T_i - T_0)/\ln \rho_i$$

and thus the shape factor is defined as

$$Q/[Lk(T_i - T_0)] = -2\pi/\ln \rho_i$$

in the W-plane which becomes, through the transformation equations,

$$Q/[Lk(T_i - T_0)] = -n\pi/\ln \rho_i \qquad (2)$$

in the Z-plane.

STEADY TRANSVERSE HEAT FLOW IN CIRCULAR RODS

In order to transform the Z-plane such as Fig. 1 into the W-plane two intermediate planes τ and τ_1 are introduced (Fig. 2). The W-plane is mapped into the first intermediate plane τ by the mapping function

$$\tau = \tfrac{1}{2} f[W/\rho_i - \rho_i/W].$$

This equation transforms the inside of the unit circle ρ_0 in the W-plane into the inside of an ellipse in the τ-plane. The circle ρ_i becomes the focal slit $2f$ in the τ-plane.

A second intermediate plane τ_1, is introduced by the mapping function

$$\tau_1 = \tau \exp[i(\beta - \pi/2)]$$

the angle β is illustrated in Fig. 2. This function reproduces the elliptic configuration of the τ-plane with the co-ordinates now rotated through an angle $(\pi/2 - \beta)$ from their position in the τ-plane.

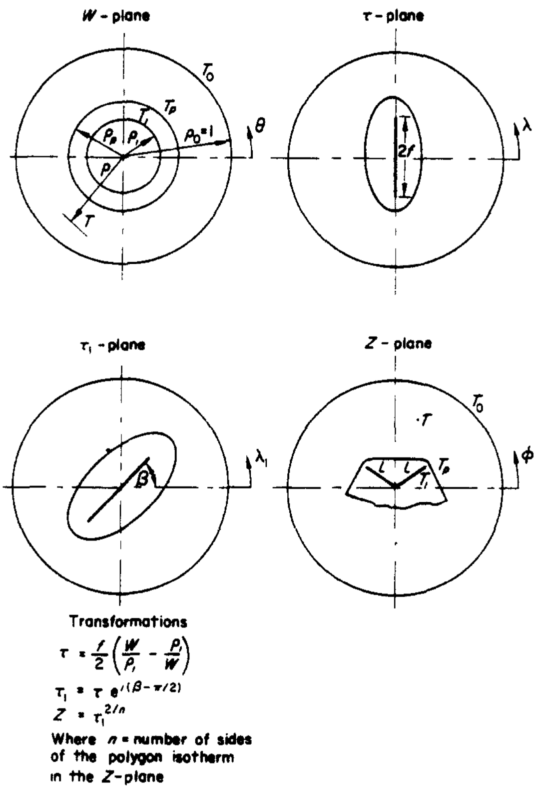


FIG. 2. Mapping planes.

Finally the function

$$Z = \tau_1^{2/n}$$

maps the confocal slits of the τ_1-plane into n equi-pitched radial slits each of length l issuing from the origin of the Z-plane. Here n also represents the number of sides of the regular polygon, formed in the Z-plane, for a particular

(ρ_p) value of ρ, i.e. the polygon is an isotherm. The radius ρ_p in the W-plane is related to the dimensions of the bar in the Z-plane through the equations

$$\rho_p/\rho_i = [(\sqrt{n}) + \sqrt{(n-2)}]/\sqrt{2} \quad (3)$$

and

$$1/\rho_i = 2(D/2l)^{n/2}. \quad (4)$$

Remote from the origin the isotherms become circular.

The shape factor for n equi-pitched narrow slits each of length l when radiating from the axis of a circular bar of diameter D is now obtained through equations (2) and (4).

$$Q/[Lk/T_i - T_0)] = 2\pi/\ln [2^{(2-n)/n} . D/l] = 2\pi/\ln [A_1 D/l]. \quad (5)$$

Values of the coefficient A_1 for various values of n are given in Table 2.

Table 2. Circular bar shape factors: values of the coefficients A_1 and A_2 for a range of values of n

n	Radial slit coefficient A_1	Polygon hole coefficient A_2
2	1·00000	...
3	0·79521	0·81226
4	0·70705	0·91018
5	0·66033	0·94711
6	0·62901	0·96512
7	0·61001	0·97527
8	0·59516	0·98155
9	0·58291	0·98570
10	0·57414	0·98860
...	...	...
...	...	...
∞	0·50000	1·00000

For the special case of $n = 2$, equation (5) becomes

$$Q/[Lk(T_i - T_0)] = 2\pi/\ln (D/l)$$

in agreement with Balcerzak and Raynor [1]. When $n = \infty$ the equation reduces to the well-known expression for flow from a central hole diameter $d(= 2l)$

$$Q/[Lk(T_i - T_0)] = 2\pi/\ln (D/d).$$

The heat flow across the isotherm ρ_p constituting the boundary of the polygon-shaped central hole in the Z-plane where the temperature is T_p is expressed by

$$Q/[Lk(T_p - T_0)] = -n\pi/\ln \rho_p.$$

The polygon is conveniently represented by the diameter d of its inscribed circle. This is related to the length l of the corresponding radial slits by

$$d = 2l[\tfrac{1}{2}(n-2)]^{1/n}.$$

These equations in conjunction with equations (3) and (4) yield an expression for the shape factor for a circular bar with a polygon-shaped central hole of n sides:

$$Q/[Lk(T_p - T_0)] = 2\pi/\ln \{[(\sqrt{n}) - \sqrt{(n-2)}]^{2/n} \times (n-2)^{1/n} . D/d\} = 2\pi/\ln [A_2 D/d]. \quad (6)$$

Values of the coefficient A_2 are given in Table 2.

The shape factors for a round rod with either a square or a hexagonal central hole were determined by Murthy and Ramachandran [2] using an electric analogue. From the graphical data empirical equations for the shape factors (written here with the notation of this paper) were defined as

$$Q/[Lk(T_p - T_0)] = 6{\cdot}533/[\ln (D/d) - 0{\cdot}15921]$$

for the round rod with a concentric square hole and

$$Q/[Lk(T_p - T_0)] = 6{\cdot}46/[\ln (D/d) - 0{\cdot}03147]$$

for the round rod with a concentric hexagonal hole.

It is interesting to notice the close similarity of the equation (6) as re-written in the form

$$Q/[Lk(T_p - T_0)] = 6{\cdot}28318/[\ln (D/d) - 0{\cdot}09411]$$

for a concentric square hole and

$$Q/[Lk(T_p - T_0)] = 6{\cdot}28318/[\ln (D/d) - 0{\cdot}03550]$$

for a concentric hexagonal hole. The agreement between the analysis and the experimental findings is excellent.

STEADY TRANSVERSE HEAT FLOW IN SEMI-INFINITE SLABS

The subsequent analysis is based on the contention that if, in a field generated between a small central hole and a semi-infinite slab, there exists a circular isothermal line, the solution for the heat flow may be represented by combining the inner region of a circular boundary configuration and the outer region of a configuration consisting of a circular hole centrally situated within a semi-infinite slab, provided that the flow lines across the common boundary (the circular isothermal line) are continuous—Fig. 3. The validity of the contention is now examined.

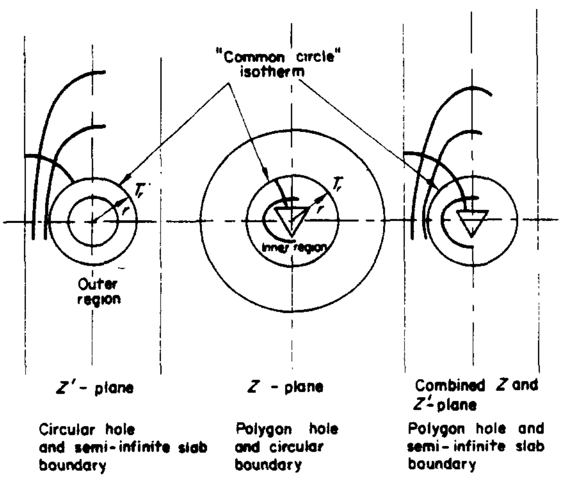


FIG. 3. Combination of the Z- and Z'-planes.

The inside of a slab with a central circular hole (Fig. 4a) may be described in the complex Z'-plane in terms of polar co-ordinates

$$Z' = \gamma' \mathrm{e}^{i\phi'}.$$

The associated complex W'-plane between two concentric circular boundaries ρ_0' and ρ_i' is expressed by

$$W' = \rho' \mathrm{e}^{i\theta'}.$$

The transformation of the inside of the unit circle ρ_0' to the inside of the slab is effected by the mapping function

$$\frac{W' - 1}{W' + 1} = i\mathrm{e}^{iZ'}.$$

This transformation was introduced by Raynor and Charnes [4] who showed that small values of ρ', e.g. the circular isothermal line ρ_r' in the W'-plane, correspond to circular isothermal lines, e.g. r in the Z'-plane. They are related by

$$\rho_r' = \pi r/2B \qquad (7)$$

where B is the thickness of the slab.

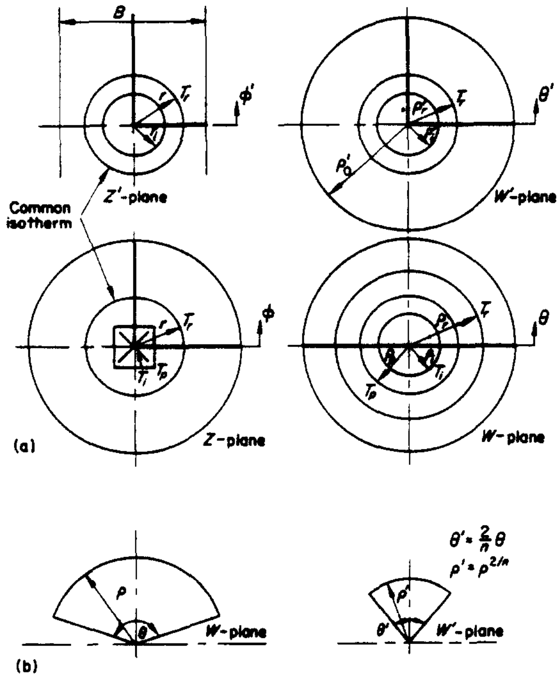


FIG. 4. Semi-infinite slab.
(a) Semi-infinite slab: mapping planes for a square hole.
(b) Relationship between the W- and W'-planes for the same flow.

In the Z-plane, since isothermal lines remote from the central polygon hole are circular, i.e. large ρ_r/ρ_i and ρ_r/ρ_p, it becomes feasible, for small holes, to combine the inner region of the Z-plane with the outer region of the Z'-plane via the common isotherm of radius r, provided that discontinuities in the radial flow lines across it are small. In the Z-plane it can be shown that for large ρ_r/ρ_i

$$|\partial\phi/\partial\theta| = 2/n = \text{constant},$$

and similarly in the Z'-plane for small ρ'

$$|\partial\phi'/\partial\theta'| = 1 = \text{constant}.$$

It is therefore clear that continuity of the radial flow lines will occur if the corresponding W- and W'-planes are so related that a given angular increment θ in the W-plane is represented by an increment of $2\theta/n$ in the W'-plane.

Further, to assess the flow from the Z-plane to the Z'-plane across their common circle r the corresponding W- and W'-planes must be so related that they have the same flow for their respective arcs, Fig. 4(b). For the same flow in the two planes, through equation (2)

$$-\frac{2\pi}{\ln \rho_i}\cdot\frac{\theta}{2\pi} = -\frac{2\pi}{\ln \rho_i'}\cdot\frac{2\theta/n}{2\pi},$$

i.e.

$$\rho_i' = \rho_i^{2/n}. \tag{8}$$

Similarly

$$\rho_r'/\rho_i' = (\rho_r/\rho_i)^{2/n} = 2^{2/n}\, r/l \tag{9}$$

for $\rho_r' \gg \rho_i'$.

Introducing equation (7) for ρ_r' into equation (9) there follows

$$\frac{1}{\rho_i'} = \left[\frac{2^{(2+n)/n}}{\pi}\right]\frac{B}{l} \tag{10}$$

and

$$\frac{Q}{Lk(T_i - T_0)} = -\frac{2\pi}{\ln \rho_i'} = 2\pi/\ln\left\{\left[\frac{2^{(2+n)/n}}{\pi}\right]\frac{B}{l}\right\} \tag{11}$$

which is the equation for the flow from n equi-length slits radiating at constant intervals from the centre of a semi-infinite slab.

For the special case of $n = \infty$, equation (11) will represent the flow from a central circular hole diameter $d(= 2l)$

$$\frac{Q}{Lk(T_i - T_0)} = 2\pi/\ln\left(\frac{4B}{\pi d}\right) \tag{12}$$

in agreement with Balcerzak and Raynor [3]. When $n = 2$ equation (11) becomes

$$\frac{Q}{Lk(T_i - T_0)} = 2\pi/\ln\left(\frac{4B}{\pi l}\right). \tag{13}$$

It can be shown that the flow based on this equation is in agreement with that obtained from Hirano's analysis [5] for the analogous flow of oil through a uniformly thin film with similar boundaries. For intermediate values of n agreement may be demonstrated with electric analogue values.

It follows from equation (3) and (8) that

$$\frac{\rho_{p'}}{\rho_i'} = \left(\frac{\rho_p}{\rho_i}\right)^{2/n} = \left[\frac{\sqrt{n} + \sqrt{(n-2)}}{\sqrt{2}}\right]^{2/n}.$$

Substituting for ρ_i' from equation (10) and for l there follows

$$\frac{Q}{Lk[T_p - T_0]} = -\frac{2\pi}{\ln(\rho_p')}$$
$$= 2\pi/\ln\left\{\frac{4B}{\pi d}\left[\frac{2}{\sqrt{[n/(n-2)]}+1}\right]^{2/n}\right\} \tag{14}$$

which is the equation for the shape factor for an n-sided polygon hole centrally situated in a slab of thickness B.

It should be noted that the orientation of the polygon relative to the slab does not affect the shape factor for small polygons, i.e. for polygons that generate an isothermal line that is circular.

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Résumé—Les facteurs de forme décrivant le régime permanent et bidimensionnel de conduction de la chaleur dans des barreux circulaires et des plaques avec différentes géométries internes et différentes conditions aux limites isothermes sont calculés par une méthode de transformation conforme.

Zusammenfassung—Formfaktoren für die stationäre, zweidimensionale Wärmeleitung in zylindrischen Stäben und Platten mit unterschiedlicher innerer Geometrie und isothermen Berandungen wurden mit Hilfe der konformen Abbildung berechnet.

Аннотация—С помощью конформного отображения рассчитаны коэффициенты формы, описывающие стационарный двумерный тепловой поток в круговых стержнях и плитах с различной внутренней геометрией при изотермических граничных условиях.